Encoding Hamiltonian circuits into multiplicative linear logic

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Abstract

We give a new proof of the NP-completeness of multiplicative linear logic without constants by a direct encoding of the hamiltonian circuit decision problem.

1 Introduction

Max Kanovich proved the NP-completeness of various fragments of multiplicative linear logic (MLL) by an encoding of the 3-partition problem [6]. We show the NP-completeness of MLL by encoding a problem of different nature, namely a graph-theoretical decision problem. This is a reference problem of the complexity theory. Our main contribution is to realize this without the use of additives. Normally a natural encoding of the hamiltonian circuit decision problem would be in the additive fragment (MALL), but this is not satisfactory because MALL is PSPACE-complete [9]. So we use a multiplicative management of the additives. We can find a similar idea in the proof of undecidability in the second order fragment of MLL [8] obtained from the result of Y.Lafont [7] where the additives are used for zero-test. We give two proofs which justify our encoding, one using proof nets, and the other using Horn implications: we obtain an interpretation of the oriented graphs as formulas and of the paths as proofs. Since the encoding is intuitionistic and MLL is conservative over its intuitionistic fragment, our result is also valid for intuitionistic multiplicative linear logic. Our approach suggests a more general study of (the foundations of) graph theory in the context of linear logic.

2 The encoding

Let G be an *oriented graph*. This means that G is a couple (V, E) with V being a finite non-empty set and $E \subseteq V \times V$. An element of V (respectively

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E) is called a *vertex* (respectively an *edge*). The first (respectively second) projection of an edge is called its *origin* (respectively *destination*). For a vertex i (respectively j) in V, we note $\deg^+(i)$ (respectively $\deg^-(j)$) the number of edges in G with origin i (respectively destination j).

A *path* from the vertex x to the vertex y in G is a sequence of edges $e_0, e_1, \ldots, e_l$ in G such that x is the origin of e_0, for every r, $0 \leq r < l$ the destination of e_r is the origin of e_{r+1} and y is the destination of e_l. A path p in G is *hamiltonian*, if every vertex of G occurs exactly once as the origin of an edge of p. A path from the vertex x to the vertex y is a *circuit* if $x = y$.

In the following we consider graphs[1] such that for each vertex i, $\deg^+(i) \geq 1$, $\deg^-(i) \geq 1$ and such that there is a vertex i, $\deg^+(i) \geq 2$ and a vertex j, $\deg^-(i) \geq 2$.

Let O be a vertex in V. Let V^* be the set $V - \{O\}$. To every vertex i in V we associate two atomic formulae a_i and b_i. It is easy to show that the existence of a hamiltonian circuit in G is equivalent to the provability in multiplicative additive linear logic of the sequent:

$$b_O, \{a_i \multimap b_i\}_{i \in V^*}, \{(b_i \multimap a_j) \mathbin{\&} 1\}_{(i,j) \in E} \vdash a_O$$

Let k be an atomic formula, and $\mathcal{S}$ the sequent[2] of MLL

$$\{k \otimes a_i \multimap k \otimes b_i\}_{i \in V^*}, \{b_i \multimap a_j\}_{(i,j) \in E}, k \otimes a_O \multimap \otimes_{i \in V} b_i^{\delta_i^+} \vdash k \otimes b_O \multimap \otimes_{j \in V} a_j^{\delta_j^-}$$

where $\delta_i^+ = \deg^+(i) - 1$ for each vertex i and

$\delta_j^- = \deg^-(j) - 1$ for each vertex j.

Theorem 1 *There is a hamiltonian circuit in the oriented graph G if and only if the sequent $\mathcal{S}$ is provable in multiplicative linear logic.*

3 The condition is necessary

If G has a hamiltonian circuit $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$, with $v_0 = v_n = O$ then we get by induction a proof of

$$\{k \otimes a_{v_i} \multimap k \otimes b_{v_i}\}_{i \in [1, \ldots, n-1]}, \{b_{v_i} \multimap a_{v_{i+1}}\}_{i \in [0, \ldots, n-1]}, k \otimes b_{v_0} \vdash k \otimes a_{v_n}.$$

[1] The main result can be stated for graphs in general.

[2] $\otimes_{i \in V} x_i^{\delta_i}$ is equivalent to $\otimes_{i \in V'} x_i^{\delta_i}$ where V' is $\{i \in V | \delta_i \neq 0\}$. See [5] for proof-nets with constants.

If E^* is the set of edges not in the hamiltonian circuit, we get easily

$$\otimes_{i \in V} b_i^{\delta_i^+}, \{b_i \multimap a_j\}_{(i,j) \in E^*} \vdash \otimes_{j \in V} a_j^{\delta_j^-},$$

and we can finish the proof of $\mathcal{S}$ by a left and a right introduction of the linear implication.

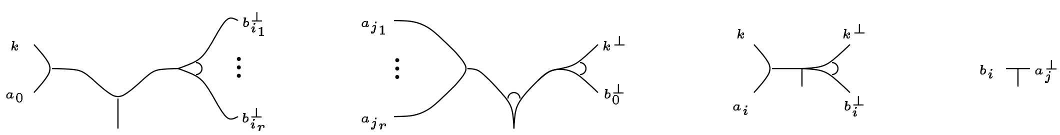
4 The condition is sufficient

4.1 Multiplicative proof-nets

We do not give full definitions for multiplicative proof-nets [5]. We use a modified version of the Danos-Regnier notation [2], and represent a binary tensor and par by: We use n-ary versions of the connectors as well. Remember

that the n-ary $⅋$ is considered as a single switch which is positioned on one of the premises. The Danos-Regnier correctness criterion for multiplicative proof-nets [2] is valid.

The following subnets, which correspond to the formulae[3] $k \otimes a_O \multimap \otimes_{i \in V} b_i^{\delta_i^+}$, $k \otimes b_O \multimap \otimes_{j \in V} a_j^{\delta_j^-}$, $k \otimes a_i \multimap k \otimes b_i$ and $b_i \multimap a_j$ are respectively called *F-device*, *I-device*, *V-device* and *E-device*[4].



4.2 Proof using proof-nets

Suppose given a proof-net $\mathcal{P}$ for the sequent $\mathcal{S}$. For a given atom A we will say that the device $\mathbf{d}_1$ is A-connected to the device $\mathbf{d}_2$ if there is an axiom-link connecting the A-port of $\mathbf{d}_1$ to the $A^\perp$-port of $\mathbf{d}_2$.

We use the correctness criterion to construct a hamiltonian circuit in G. Consider the k-axiom-links in $\mathcal{P}$. The acyclicity condition forbids the existence

[3] The formulas on the left in the sequent $\mathcal{S}$ are negated.

[4] The notations stand respectively for final, initial, vertex and edge.

of a sequence of V-devices $\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_l$, such that $\mathbf{v}_0 = \mathbf{v}_l$ and for each i, $0 \leq i < l$, $\mathbf{v}_i$ is k-connected to $\mathbf{v}_{i+1}$. It would be sufficient to put each $⅋$-switch occurring in one of the V-devices in the sequence on the position k to get a cycle. If we call $\mathbf{v}_0$ the I-device, then there is a sequence of V-devices $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$, such that for each i, $0 \leq i \leq n-1$, $\mathbf{v}_i$ is k-connected to $\mathbf{v}_{i+1}$, and $\mathbf{v}_n$ is the F-device. Every V-device $\mathbf{v}$ and the F-device is a_j-connected to an E-device.

If an E-device $\mathbf{e}$ is b_i-connected to the F-device, then the I-device is a_j-connected to $\mathbf{e}$, otherwise one would get a cycle by putting the $⅋$-switch of the F-device on the position corresponding to $\mathbf{e}$ and all other $⅋$-switches on V-devices on the k-position. From $\Sigma_{i\in V}\delta_i^+ = \Sigma_{i\in V}\delta_i^-$ we have that if the I-device is a_j-connected to an E-device $\mathbf{e}$, then $\mathbf{e}$ is b_i-connected to the F-device.

We prove by downward induction on the integer r, $r < n$ that if the V-device (or the F-device) $\mathbf{v}_{r+1}$ is a_j-connected to the E-device $\mathbf{e}_r$, and $\mathbf{e}_r$ is b_i-connected to a device $\mathbf{u}$, then $\mathbf{u}$ equals $\mathbf{v}_r$. If $\mathbf{u}$ is a $\mathbf{v}_l$, with $l < r$, by switching $\mathbf{v}_r$ on $a_j^\perp$, and $\mathbf{v}_l$ on $k^\perp$, we disconnect the proof-net. Thus $\mathbf{u}$ equals $\mathbf{v}_r$. The sequence $e_0, e_1, \dots, e_{n-1}$, where e_l is the edge corresponding to the E-device $\mathbf{e}_l$, yields a hamiltonian circuit of G.

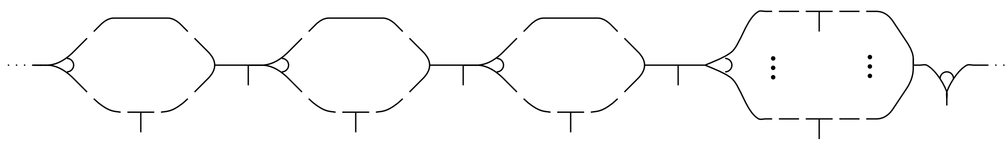


Fig. 1. Example of a proof-net

4.3 Proof using horn programs

Definition 2 *A* simple conjunction *is a tensor of positive literals.*

A Horn implication *is a formula of the form $X \multimap Y$ where X and Y are simple conjunctions.*

Definition 3 *For a multiset Γ consisting of Horn implications, a sequent of the form $W, \Gamma \vdash Z$ where W and Z are simple conjunctions is called a* Horn sequent.

Note that if $W = k \otimes b_O$, $Z = \otimes_{j\in V} a_j^{\delta_j^-}$ and

$$\Gamma = \left\{ \{(k \otimes a_i) \multimap (k \otimes b_i)\}_{i\in V^*}, \{b_i \multimap a_j\}_{(i,j)\in E}, (k \otimes a_O) \multimap \otimes_{i\in V} b_i^{\delta_i^+} \right\}$$

then $W, \Gamma \vdash Z$ is a Horn sequent. By reversibility of right linear implication, it is provable if and only if $\mathcal{S}$ is provable.

The idea of M.Kanovich [6] is that a branching Horn program produces Z from W by consuming generalized Horn implications of Γ. Because our Γ is a multiset consisting only of Horn implications, we use a restricted form of Horn programs and suitable theorems.

Definition 4 *A* Horn program *is a chain where each vertex is labelled by a simple conjuction and each edge is labelled by a Horn implication $X \multimap Y$ which describes the elementary assignment operation producing $Y \otimes U$ from $X \otimes U$.*

Theorem 5 (Completeness[6]) *For any Γ consisting of Horn implications, a sequent of the form*

$$W, \Gamma \vdash Z$$

is derivable in linear logic if and only if we can construct a Horn program P such that

(1) All formulas used in the program P are from Γ,
(2) In the chain P each formula of Γ is used exactly once,
(3) The first node is labelled by W and the last one by Z.

If the sequent $\mathcal{S}$ is provable then by the completeness theorem we can construct a Horn program P which satisfies:

- P starts from $k \otimes b_O$ and reaches $k \otimes a_O$ at a certain node using alternatively formulae of type $b_i \multimap a_j$ and $(k \otimes a_i) \multimap (k \otimes b_i)$:

$$\begin{tikzcd}[row sep=large]
\boxed{n_l} \arrow[d, "b_i \multimap a_j"] & k \otimes b_i \\
\boxed{n_{l+1}} \arrow[d, "(k \otimes a_j) \multimap (k \otimes b_j)"] & k \otimes a_j \\
\boxed{n_{l+2}} & k \otimes b_j
\end{tikzcd}$$

- All the $\{(k \otimes a_i) \multimap (k \otimes b_i)\}_{i \in V^*}$ are used before one reaches $k \otimes a_O$.

Here is the key point of the proof: if a node in P has the label $k \otimes a_j$ then the next edge cannot have the labelling $(k \otimes a_O) \multimap \otimes_{i \in V} b_i^{\delta_i^+}$ before we have already used all of the $\{(k \otimes a_i) \multimap (k \otimes b_i)\}_{i \in V^*}$. Otherwise the next node has the label $\otimes_{i \in V} b_i^{\delta_i^+}$ which does not contain an occurence of k and then no following edge in the chain can be labelled by a $(k \otimes a_i) \multimap (k \otimes b_i)$. This contradicts the fact that in the chain P each formula of Γ is used exactly once. So this implies the existence of a hamiltonian circuit in $G = (V, E)$.

In this section we work with proofs in intuitionistic sequent calculus.

Lemma 6 *Let $U \subseteq V$ and $E \subseteq V \times V$.*

i) If $k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U}, \{b_p \multimap a_q\}_{(p,q) \in E} \vdash k \otimes a_j$ is provable with $\{i, j\} \notin U$ then there exists a hamiltonian path from i to j in $G = (U \cup \{i, j\}, E)$.

ii) If $k, a_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U}, \{b_p \multimap a_q\}_{(p,q) \in E} \vdash k \otimes a_j$ is provable with $i \in U$ and $j \notin U$ then there exists a hamiltonian path from i to j in $G = (U \cup \{j\}, E)$,

iii) If $k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U}, \{b_p \multimap a_q\}_{(p,q) \in E} \vdash k \otimes b_j$ is provable with $i \notin U$ and $j \in U$ then there exists a hamiltonian path from i to j in $G = (U \cup \{i\}, E)$,

iv) If $k, a_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U}, \{b_p \multimap a_q\}_{(p,q) \in E} \vdash k \otimes b_j$ is provable with $\{i, j\} \in U$ then there exists a hamiltonian path from i to j in $G = (U, E)$.

PROOF. By induction on $n = card(U) + card(E)$. Let $P(n)$ the conjunction of i) to iv) at rank n. Suppose that $P(m)$ is true for all $m < n$.

Case i): if $k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U}, \{b_p \multimap a_q\}_{(p,q) \in E} \vdash k \otimes a_j$ is provable then consider the last rule in a cut-free proof of this sequent:

- rule of left linear implication on $k \otimes a_l \multimap k \otimes b_l$ for $l \in U$. Balance of atoms implies that the first sequent is provable if and only if

$$\left\{ \begin{array}{c} k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_1}, \{b_p \multimap a_q\}_{(p,q) \in E_1} \vdash k \otimes a_l \\ k \otimes b_l, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_2}, \{b_p \multimap a_q\}_{(p,q) \in E_2} \vdash k \otimes a_j \end{array} \right.$$

  are provable where $\{U_1, U_2\}$ is a partition of $U \backslash \{l\}$ and $\{E_1, E_2\}$ is a partition of E. By reversibility of left tensor rule and induction hypothesis i) there are hamiltonian paths from i to l in $G = (U_1 \cup \{i, l\}, E_1)$ and from l to j in $G = (U_2 \cup \{l, j\}, E_2)$. Because $\{i, j\} \notin U$, there exists a hamiltonian path from i to j in $G = (U_1 \cup U_2 \cup \{l, i, j\}, E_1 \cup E_2)$ i.e. in $G = (U \cup \{i, j\}, E)$.

- rule of left linear implication on $b_r \multimap a_s$ for $(r, s) \in E$. Balance of atoms implies that the first sequent is provable if and only if

$$\left\{ \begin{array}{cr} b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_1}, \{b_p \multimap a_q\}_{(p,q) \in E_1} \vdash b_r & (1) \\ k, a_s, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_2}, \{b_p \multimap a_q\}_{(p,q) \in E_2} \vdash k \otimes a_j & (2) \end{array} \right.$$

are provable where $\{U_1, U_2\}$ is a partition of U and $\{E_1, E_2\}$ is a partition of $E\backslash\{(r,s)\}$. By case analysis of the last rule, (1) is provable if and only if $i = r$ and $U_1 = E_1 = \emptyset$. By induction hypothesis ii) on (2), there is a hamiltonian path from s to j in $G = (U_2 \cup \{j\}, E_2)$. Because $i \notin U$, there exists a hamiltonian path from i to j in $G = (U_2 \cup \{i,j\}, E_2 \cup \{(r,s)\})$.

- rule of right tensor on $k \otimes a_j$. Balance of atoms implies that the first sequent is provable if and only if

$$\begin{cases} k, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_1}, \{b_p \multimap a_q\}_{(p,q) \in E_1} \vdash k & (1) \\ b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_2}, \{b_p \multimap a_q\}_{(p,q) \in E_2} \vdash a_j & (2) \end{cases}$$

  are provable where $\{U_1, U_2\}$ is a partition of U and $\{E_1, E_2\}$ is a partition of E. It follows from a study of the last rule that (1) is provable if and only if $U_1 = E_1 = \emptyset$. Likewise (2) is provable if and only if $U_2 = \emptyset$ and $E_2 = \{(i,j)\}$ (*i.e.* $n = 1$). $G = (\{i,j\}, E_2)$ has a trivial hamiltonian path from i to j.

Case ii) is similar to case i) except that the last rule in a cut-free proof of this sequent can be a rule of right tensor on $k \otimes a_j$ if and only if $i = j$ and $U = E = \emptyset$ (*i.e.* $n = 0$). But also it cannot be a rule of left linear implication on $b_r \multimap a_s$ for $(r,s) \in E$ because atoms cannot be balanced.

Case iii) is similar to case i).

Case iv) is similar to case i) except that the last rule in a cut-free proof of this sequent cannot be a rule of left linear implication on $b_r \multimap a_s$ for $(r,s) \in E$ or a rule of right tensor on $k \otimes b_j$ because atoms cannot be balanced.

So $P(n)$ is true. $\square$

Lemma 7 *Let $U \subseteq V$ and $E \subseteq V \times V$.*

i) If $k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U^*}, \{b_p \multimap a_q\}_{(p,q) \in E}, (k \otimes a_O) \multimap \otimes_{i \in U} b_i^{\delta_i^+} \vdash \otimes_{j \in U} a_j^{\delta_j^-}$ is provable with $\{i\} \notin U$ then there exists a hamiltonian path from i to O in $G = (U \cup \{i, O\}, E)$,

ii) If $k, a_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U^*}, \{b_p \multimap a_q\}_{(p,q) \in E}, (k \otimes a_O) \multimap \otimes_{i \in U} b_i^{\delta_i^+} \vdash \otimes_{j \in U} a_j^{\delta_j^-}$ is provable with $i \in U$ then there exists a hamiltonian path from i to O in $G = (U \cup \{O\}, E)$.

PROOF. By induction on $n = card(U) + card(E)$. Let $P(n)$ be i) and ii) at rank n. Suppose that $P(m)$ is true for all $m < n$.

Case i): if $k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U^*}, \{b_p \multimap a_q\}_{(p,q) \in E}, (k \otimes a_O) \multimap \otimes_{i \in U} b_i^{\delta_i^+} \vdash \otimes_{j \in U} a_j^{\delta_j^-}$ is provable then consider the last rule in a cut-free proof of this sequent:

- rule of left linear implication on $k \otimes a_l \multimap k \otimes b_l$ for $l \in U^*$ and rule of left linear implication on $b_r \multimap a_s$ for $(r, s) \in E$. Similar to case i) of lemma 6, using reversibility of left tensor rule, induction hypothesis and lemma 6.
- rule of left linear implication on $(k \otimes a_O) \multimap \otimes_{i \in U} b_i^{\delta_i^+}$. Balance of atoms implies that the first sequent is provable if and only if

$$\begin{cases} \otimes_{i \in U} b_i^{\delta_i^+}, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_1}, \{b_p \multimap a_q\}_{(p,q) \in E_1} \vdash \otimes_{j \in U} a_j^{\delta_j^-} & (1) \\ k, b_i, \{k \otimes a_p \multimap k \otimes b_p\}_{p \in U_2}, \{b_p \multimap a_q\}_{(p,q) \in E_2} \vdash k \otimes a_O & (2) \end{cases}$$

  are provable where $\{U_1, U_2\}$ is a partition of U^* and $\{E_1, E_2\}$ is a partition of E. By case analysis of the last rule, (1) is provable if and only if $U_1 = \emptyset$. Then $E_1 \subseteq U \times U$. By lemma 6 i) on (2) there is a hamiltonian path from i to O in $G = (U_2 \cup \{i, O\}, E_2)$. So there exists a hamiltonian path from i to O in $G = (U_2 \cup \{i, O\}, E_1 \cup E_2)$.
- rule of right tensor on $\otimes_{j \in U} a_j^{\delta_j^-}$ cannot appear by balance of atoms and considering possible rules. In fact a particular study is needed if the number of edges is two more than the number of vertices.

Case ii) is the same as case i) except that the last rule in a cut-free proof of this sequent cannot be a rule of left linear implication on $b_r \multimap a_s$ for $(r, s) \in E$ because atoms cannot be balanced.

So $P(n)$ is true. $\square$

PROOF (encoding provable $\Rightarrow$ existence of a hamiltonian circuit). By reversibility of the right linear implication and of the left tensor rule, provability of $\mathcal{S}$ implies that the hypothesis of lemma 7 i) with $U = V$ is satisfied. So there is a hamiltonian path from O to O in $G = (V^* \cup O, E)$ *i.e.* there exists a hamiltonian circuit in $G = (V, E)$. $\square$

5 Conclusion

The encoding should give some intuition for a multiplicative management of additives in other cases as well. We remark that we use a small fragment of multiplicative linear logic. In fact with a slight modification of the encoding the valid proof-nets are planar. The question of NP-completeness of noncommutative multiplicative linear logic[5] remains open though, as no order is imposed *a priori* on the formulae of the sequent $\mathcal{S}$.

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[5] see [10] for a partial result.

6 Appendix

6.1 Sequent calculus for intuitionistic multiplicative linear logic

A formula is either a positive atom A, or a negative one $A^\perp$, or a constant 1, or constructed using binary connectors $A \otimes B$ (tensor), $A \multimap B$ (linear implication). Intuitionistic sequents are of the form $\Gamma \vdash A$ where Γ is a multiset of formulae and A a formula. The rules for the intuitionistic sequent calculus are the following:

Identity group

$$\frac{}{A \vdash A}\ \text{(identity)} \qquad \frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}\ \text{(cut)}$$

Logic group

unit:

$$\frac{}{\vdash 1}\ \text{(one)}$$

tensor:

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \text{(left)} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \text{(right)}$$

linear implication:

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\ \text{(left)} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\ \text{(right)}$$

6.2 Some properties

- Classical multiplicative linear logic is conservative over intuitionistic multiplicative linear logic (see [4] for definitions): an intuitionistic sequent is provable in the intuitionistic calculus if and only if it is classically provable.
- The calculus verifies cut elimination, so a provable sequent has a proof not using the cut rule.
- A rule is *reversible* if the provability of its conclusion implies the provability of its premises. The left tensor rule and the right linear implication rule are reversible.
- Balance of atoms: if we define $p_A(A) = 1, p_A(B \otimes C) = p_A(B) + p_A(C)$ and $p_A(B \multimap C) = p_A(C) - p_A(B)$ for an atom A then every provable sequent $B_1, \ldots, B_n \vdash C$ satisfies $p_A(B_1) + \ldots + p_A(B_n) = p_A(C)$.